

The tsunami mode of a flat earth and its excitation by earthquake sources

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Summary. Tsunami generation by earthquakes in a flat, isotropic, elastic, vertically stratified earth underlying a uniform-depth, incompressible ocean can be studied in terms of the tsunami normal mode of the combined ocean–solid earth system. We derive, in a way that demonstrates their natural extension from traditional approaches to tsunami theory, the equations and boundary conditions governing the tsunami mode displacement and stress eigenfunctions, then solve the excitation problem by a variational method. This leads to a straightforward expression for the far-field tsunami displacements due to a point moment tensor source in the solid earth. Numerically computed spectra and waveforms reveal clearly the dependence of the far-field tsunami on the source depth, duration, moment and mechanism.

Introduction

A central problem in tsunami generation is to determine the waves excited by a realistic earthquake source buried in the solid earth underneath the ocean. Early work on tsunami generation frequently involved some time-dependent deformation, circular, elliptical or rectangular in shape, driving an ocean which overlay an otherwise rigid bottom (e.g. Takahashi 1942; Momoi 1962; Kajiura 1963). A few authors have extended this approach to use more realistic, static vertical deformation patterns due to earthquake fault models and observations, rather than simple but arbitrary shapes and patterns; Ando (1982) provides a good review. The correctness of even the latter models is not readily apparent, however, since the time-dependence of the deformation is somewhat arbitrary and the full interaction of the ocean and solid earth is approximated by a partial decoupling in which the solid earth can drive the ocean but no reciprocal action is permitted.

One way to incorporate the full ocean–earth interaction into a model of tsunami generation by a realistic earthquake source is to use normal mode theory similar to that applied to seismic surface waves. Tsunamis are indeed guided surface waves, closely akin to Rayleigh waves. They differ most significantly in that the restoring force is gravitational in one case and elastic in the other, and that tsunamis consist of only one mode, corresponding

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to gravity waves on the ocean surface. Therefore, in this paper we derive the equations and boundary conditions governing the tsunami mode of a flat, uniform-depth ocean overlying an elastic isotropic solid earth with elastic parameters and density varying only with depth. The far-field tsunami excited by a point moment tensor source is then obtained in terms of the normal mode eigenfunctions by a variational technique extended from the one developed by Saito (1967) for seismic surface wave excitation.

Several other investigators have attacked the problem of tsunami generation with the ocean and solid earth fully coupled. Podyapolsky (1970) and Alexeev & Gusiakov (1976) considered a point earthquake source in the solid earth, which was represented by an elastic half-space; however, the accounts of their methods are rather incomplete and only a few results are illustrated. Yamashita & Sato (1974) extended a similar model to a finite, moving source, but the explanation of how they perform the key step of evaluating the residue at the 'tsunami pole' is omitted. And although the works just cited are all based on flat earth models, none make reference to the tsunami normal mode of a flat ocean–earth system. Ward (1980, 1981, 1982a, b) has introduced normal modes in the tsunami generation problem, but in the context of a spherically symmetric ocean–earth model. He formulated the equations of motion in a manner originally applied to the Earth's free oscillations and made use of a very general result to obtain the tsunami mode excitation.

Ward's results are useful and important, yet it is nonetheless also rewarding to explore tsunami normal mode excitation using a flat earth model. At the very least, two independent solutions of very similar problems can be used to check one another. Also, no significant increase in accuracy can be obtained simply by going from a flat earth model to a spherical one, since the tsunami mode eigenfunctions (unlike those of long-period seismic surface waves) do not penetrate the solid earth very deeply, and a correction for geometric spreading on a spherical, rather than flat, surface is easily applied. Of course, the geometric spreading of real tsunamis is different from either idealized case, due to the bathymetric variations (resulting in variations in wave speed) in the real oceans.

A more substantial difference between the present work and Ward's (1980) is that we consider only the most important forces. Elastic, gravitational and inertial terms, and the effects of self-gravitation, are all included by Ward in both the solid earth and ocean layer. However, tsunamis are basically ocean surface gravity waves and elastic terms in the ocean are of very secondary importance. Lighthill (1978), for example, notes that acoustic and gravity waves are fully decoupled in the oceans and Stoneley (1963) demonstrated that modelling the ocean as an incompressible fluid is a very good approximation with respect to tsunami propagation. Ward's exploration of the energy partitioning in tsunami modes confirms this: the gravitational and inertial forces are of large and roughly equal significance in the ocean, and also, in the solid earth elastic forces are much more important than inertia or gravity. Actually, in terms of tsunami *propagation*, displacements of the solid earth can be neglected altogether, which has been the traditional approach in tsunami studies, yet they are crucial in terms of tsunami mode *excitation* by sources within the solid earth.

Hence we now consider surface gravity waves on an incompressible fluid in a uniform gravitational field overlying an elastic earth in which gravitational forces are ignored and for which it will be shown that elastic forces are far more important than inertia. A side benefit is that we are able to build a conceptual bridge between the traditional approach to tsunami modelling based on the hydrodynamics of an incompressible, flat ocean (e.g. Kajiura 1963) and the normal mode methods. We first develop the equations governing the tsunami mode, then study its excitation. We note here that the flat earth excitation problem differs substantially from the corresponding spherical earth problem. There is a continuum of frequencies and wavenumbers for a flat earth but frequencies and angular orders are discrete on a spherical earth. Also, the normal modes of a finite body form a complete basis for the

small amplitude oscillations of the body but those of an infinite body, like the flat earth, do not. Consequently the theoretical development of the solution to the excitation problem given here is totally distinct from Ward's (1980). Finally, we remark that, in common with Ward, we assume the ocean to be inviscid and deal only with linearized equations of motion and boundary conditions. The former assumption was justified by Stoneley (1963) and linearity, which is usually assumed in seismology but often not in hydrodynamics, is considered in the following, brief section.

Linearity

Exact hydrodynamic equations are non-linear, due to both convective inertia terms and boundary conditions. However, it is widely agreed that non-linear effects are unimportant in the generation and deep ocean propagation stages of a tsunami and can become important only in its coastal interactions (Carrier 1971; Le Méhauté 1976; Wu 1979). Hammack (1973) and Hammack & Segur (1978) found that for one-dimensional propagation non-linear terms become important after a certain time, which they established clearly, but, as Wu (1979) pointed out, in two-dimensional propagation the decrease in amplitude due to geometric spreading eliminates this effect.

One clear condition for the linearity of waves of long wavelength λ is that the amplitude η_0 be small compared to the water depth h . That is,

$$\eta_0/h \ll 1$$

(In fact it follows from the results of the next section that for small h/λ the non-linear term $\frac{1}{2}v^2$ in the Bernoulli equation (8) is much smaller than the term $\partial\phi/\partial t$ if this condition is satisfied.) Another widely posed criterion for the linearity of long waves is whether the Ursell parameter $\eta_0\lambda^2/h^3$ is small with respect to 1 (Ursell 1953; Le Méhauté 1976). Generally the Ursell parameter is not too large but can be of order 1 near the origin of a tsunami (Hammack & Segur 1978; Wu 1979). However, the Ursell parameter is essentially a ratio of the relative importance of non-linear effects to linear dispersive effects for long waves and if both effects are small that ratio may not be significant. For example, the use of linear dispersive theory should be correct as long as non-linear effects are small, even if dispersion is negligible.

Indeed, linear non-dispersive theory, based on the small amplitude condition and the condition

$$h/\lambda < 0.05$$

is often applied to tsunamis (e.g. Ando 1982) and is valid for large wavelength sources and short propagation paths. In this case all waves travel with the speed $\sqrt{gh}$, where g is the acceleration of gravity. An improvement on this is the linear Boussinesq theory in which a small correction, good at long and moderate wavelengths, is applied to account for dispersion (e.g. Carrier 1971).

Throughout this paper the long-wavelength assumption is avoided, however, and we use linear dispersive hydrodynamic theory valid for all wavelengths. This is advantageous in treating point sources which have more short-wavelength radiation than real tsunamis and, of course, the validity of linear superposition makes powerful and straightforward methods readily applicable.

The tsunami mode

We consider plane water waves propagating across a flat, ideal, incompressible ocean of uniform depth h overlying a solid isotropic earth whose elastic parameters and density vary

only with depth. Fig. 1 illustrates the geometry and coordinate system chosen for this problem. A constant gravitational field with acceleration g acts in the downward (positive z) direction in the fluid, and is ignored in the solid. The x - and z -components of the Lagrangian displacement field may be assumed to have the form

$$\begin{aligned} u &= r_1(z) \cos(kx - \omega t) \\ w &= -r_2(z) \sin(kx - \omega t), \end{aligned} \quad (1)$$

where t denotes time, k the wavenumber, and ω the angular frequency, corresponding to propagation in the positive x -direction. r_1 and r_2 , functions only of z , are the displacement eigenfunctions of the tsunami mode. The vertical stress components due to the wave can similarly be expressed as

$$\begin{aligned} \sigma_{zx} &= r_3(z) \cos(kx - \omega t) \\ \sigma_{zz} &= -r_4(z) \sin(kx - \omega t). \end{aligned} \quad (2)$$

We next determine the systems of equations, in both solid and fluid, and the boundary conditions to be satisfied by the displacement-stress vector $\mathbf{r} = (r_1, r_2, r_3, r_4)$.

Our approach so far is identical to the displacement-stress vector formulation for the Rayleigh wave modes of a flat, vertically stratified earth. This is not surprising, since both wave types involve particle motion in a vertical plane parallel to the propagation direction. Therefore, for convenience, we adopt the notation which Aki & Richards (1980) apply to Rayleigh waves and, whenever it is helpful, we refer directly to their equation numbers, prefacing each with 'AR'.

In the solid earth, just as in the Rayleigh-wave case, the tsunami displacement-stress vector satisfies (AR7.28), or

$$\frac{d}{dz} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} 0 & k & \mu^{-1} & 0 \\ -k\lambda[\lambda + 2\mu]^{-1} & 0 & 0 & [\lambda + 2\mu]^{-1} \\ k^2\xi - \omega^2\rho_s & 0 & 0 & k\lambda[\lambda + 2\mu]^{-1} \\ 0 & -\omega^2\rho_s & -k & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \quad (3)$$

where $\xi = 4\mu(\lambda + \mu)/(\lambda + 2\mu)$ and the Lamé parameters λ and μ and density, ρ_s are functions of z . Because gravity is neglected in (3), it is not possible to obtain the corresponding system of equations for the incompressible ocean layer simply by taking the limits $\mu \rightarrow 0$, $\lambda \rightarrow \infty$.

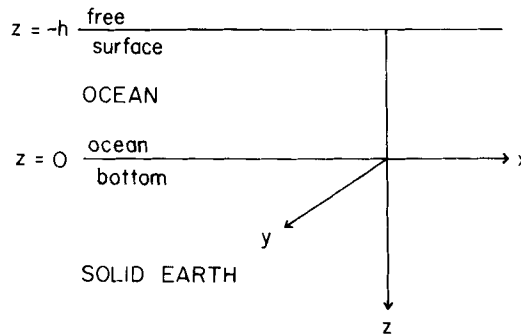


Figure 1. Geometry and coordinate system for tsunami calculations.

Instead, we derived the equations governing $\mathbf{r}$ in the fluid through a very classical formulation (e.g. Officer 1974) beginning with a scalar velocity potential.

Assuming irrotational motion, we introduce a scalar velocity potential ϕ such that the velocity field of the fluid is given by

$$\mathbf{v} = -\nabla\phi. \quad (4)$$

Since the fluid is incompressible, the equation of continuity is simply

$$\nabla^2\phi = 0. \quad (5)$$

We also introduce the vertical displacement of the free surface which we denote by η , where η is positive for upward displacement (in the *negative* z -direction). Naturally,

$$\eta(x, y, t) = -w(x, y, -h, t). \quad (6)$$

Actually (6) is approximate in two ways: (1) it is linearized in η with the right-hand side evaluated at $z = -h$ rather than $z = -h - \eta$ and (2) η is an Eulerian displacement, but since the wave amplitudes are small and the waves taken to be periodic, the displacements are small and the distinction between the Lagrangian and Eulerian approaches can be neglected.

Another important quantity is the pressure p in the fluid. At the free surface it must equal atmospheric pressure p_0 , or

$$p(x, y, -h - \eta, t) = p_0. \quad (7)$$

The Bernoulli equation for an incompressible fluid gives a general expression for the pressure in the fluid:

$$\frac{p}{\rho} = \frac{\partial\phi}{\partial t} - \Omega - \frac{1}{2}v^2 + C(t) \quad (8)$$

where ρ is the fluid density, Ω the gravitational potential, and $C(t)$ is a constant of integration with respect to spatial coordinates. Ω satisfies $\mathbf{g} = -\nabla\Omega$ where $\mathbf{g}$ is the gravitational field. We can set $\Omega = -g(z + h)$, fixing an arbitrary constant of integration. In order to satisfy (7) and (8) in the case of static equilibrium ($\partial\phi/\partial t = 0$, $v = 0$, $\eta = 0$) we need $C(t) = p_0/\rho$. Then, neglecting the non-linear term $\frac{1}{2}v^2$, (8) becomes

$$\frac{p}{\rho} = \frac{\partial\phi}{\partial t} + g(z + h) + \frac{p_0}{\rho}. \quad (9)$$

At this point we also note that from (7) and (9)

$$\eta = \left. \frac{1}{g} \frac{\partial\phi}{\partial t} \right|_{z=-h} \quad (10)$$

which has been linearized in η by taking $z = -h$ rather than $z = -h - \eta$ on the right-hand side. A second relation between η and ϕ is obtained from (6) and (4), $\partial\eta/\partial t = \partial\phi/\partial z$ at $z = -h$. Combined, these give the free surface boundary condition on the velocity potential.

$$\left(\frac{1}{g} \frac{\partial^2\phi}{\partial t^2} - \frac{\partial\phi}{\partial z} \right)_{z=-h} = 0. \quad (11)$$

We now relate the classical quantities ϕ and p to the displacement-stress vector, and derive the equations governing $\mathbf{r}$ in the fluid. From (4), $\partial\mathbf{u}/\partial t = -\partial\phi/\partial\mathbf{x}$, which together with (1) yields

$$\phi = cr_1(z) \cos(kx - \omega t), \quad (12)$$

where $c = \omega/k$ is the tsunami phase velocity. Then, from (9) and (12)

$$p = p_0 + \rho g(z + h) + \rho \frac{\omega^2}{k} r_1(z) \sin(kx - \omega t). \quad (13)$$

In the fluid the shear stress σ_{zx} must vanish and the normal stress σ_{zz} is simply equal to minus the non-static component of p , so that

$$\begin{aligned} r_3 &= 0 \\ r_4 &= \rho \frac{\omega^2}{k} r_1. \end{aligned} \quad (14)$$

Thus r_3 and r_4 are redundant in the fluid, and the analogue to (3) is a system of two first-order equations involving only r_1 and r_2 . It is readily obtained. From (4), $\partial w / \partial t = -\partial \phi / \partial z$, which leads to $dr_1/dz + kr_2 = 0$ when combined with (1) and (12). Rewriting the continuity equation (5) as $\partial u / \partial x + \partial w / \partial z = 0$, we find from (1) that $dr_2/dz + kr_1 = 0$. Hence

$$\frac{d}{dz} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 0 & -k \\ -k & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \quad (15)$$

It is straightforward to formulate the (linearized) boundary conditions on $\mathbf{r}$. At the interface $z = 0$ the vertical component of displacement w must be continuous, the shear stress σ_{zx} must vanish, and the vertical normal stress σ_{zz} must be continuous. Thus

$$r_2^{\text{solid}}(0) = r_2^{\text{fluid}}(0) \quad (16a)$$

$$r_3^{\text{solid}}(0) = r_3^{\text{fluid}}(0) = 0 \quad (16b)$$

$$r_4^{\text{solid}}(0) = r_4^{\text{fluid}}(0) = \rho \frac{\omega^2}{k} r_1^{\text{fluid}}(0). \quad (16c)$$

Superscripts ‘solid’ and ‘fluid’ are added here and wherever they are needed for clarity. The free surface boundary condition (11) can also be expressed in terms of the displacement-stress vector. With the use of (4), (1) and (12) it becomes

$$\omega^2 r_1(-h) = gkr_2(-h). \quad (17)$$

Two more boundary conditions are required in general,

$$r_1, r_2 \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty, \quad (18)$$

that is, displacements must vanish at infinite depth in the solid earth.

Equation (15) may be solved analytically; for $z < 0$

$$\begin{aligned} r_1(z) &= A \cosh kz + B \sinh kz \\ r_2(z) &= -B \cosh kz - A \sinh kz. \end{aligned} \quad (19)$$

The constants A and B and the displacement eigenfunctions $r_1(z)$ and $r_2(z)$ in the solid earth (for $z > 0$) must be obtained numerically from (3) and the boundary conditions (16), (17) and (18). Such a solution will also yield the dispersion relation for the tsunami mode, which follows from the requirement that the equations (15) and (3) and the boundary conditions, which are all homogeneous, be mutually consistent.

In the special case where the solid earth is an elastic half-space in which $\lambda = \mu = \text{constant}$,

the eigenfunctions may be approximated by the following analytic expressions: for $z < 0$ (ocean)

$$\begin{aligned} r_1(z) &= \cosh kz + (3\rho c^2/4\mu) \sinh kz \\ r_2(z) &= -\sinh kz - (3\rho c^2/4\mu) \cosh kz \end{aligned} \quad (20)$$

and for $z > 0$ (solid earth)

$$\begin{aligned} r_1(z) &= -(\rho c^2/2\mu) (kz - 1/2) \exp(-kz) \\ r_2(z) &= -(\rho c^2/2\mu) (kz + 3/2) \exp(-kz) \\ r_3(z) &= \rho c^2 k (kz) \exp(-kz) \\ r_4(z) &= \rho c^2 k (kz + 1) \exp(-kz). \end{aligned} \quad (21)$$

(Note that in these expressions r_1 and r_2 are dimensionless, while r_3 and r_4 have dimensions of stress divided by length, so that to be strictly consistent with (1) and (2) $\mathbf{r}$ must be multiplied by a constant length scale.) It is obvious that (20) satisfies (15) exactly, and that (16) and (18) are satisfied exactly by (20) and (21). The only approximation involved is that (21) satisfies (3) only if the term $\omega^2 \rho_s$ is replaced by zero. At the lower frequencies important for tsunamis this is reasonable, and corresponds physically to neglecting inertial forces with respect to elastic forces in the solid earth. The phase velocity c must of course be calculated before (20) and (21) can be evaluated. It can be obtained by numerically solving the dispersion relation that results from substituting (20) into the free surface condition (17). A phase velocity curve computed in this way, for a typical set of model parameters, is illustrated in Fig. 2.

Fig. 2 also shows that, except at very low frequencies, the phase velocity is well approximated by the well-known rigid bottom result,

$$c = [(g/k) \tanh kh]^{1/2} \quad (22)$$

which follows from requiring that (19) satisfy (17) and the condition $r_2(0) = 0$. Under this condition $B/A = 0$ in (19) and $\mathbf{r} = 0$ in the solid earth, which also follows from (20) and (21)

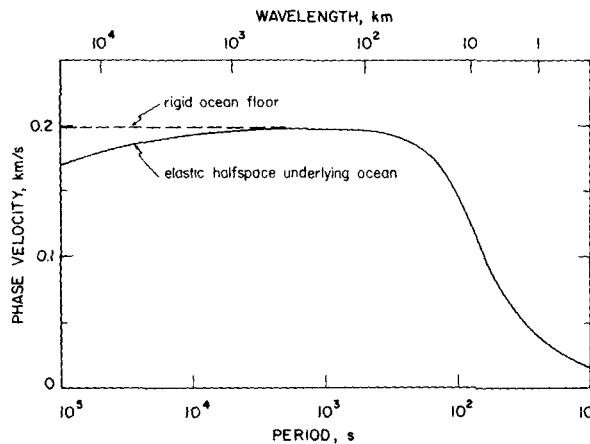


Figure 2. Tsunami phase velocities for a 4 km deep ocean. The parameters chosen for the half-space are $\rho_s = 3.1 \text{ g cm}^{-3}$ and $\lambda = \mu = 5.2 \times 10^{11} \text{ dyn cm}^{-2}$ (corresponding to P and S velocities of 7.15 and 4.1 km s^{-1} , respectively). The period scale is valid for both curves but the wavelength scale was determined for the rigid bottom case.

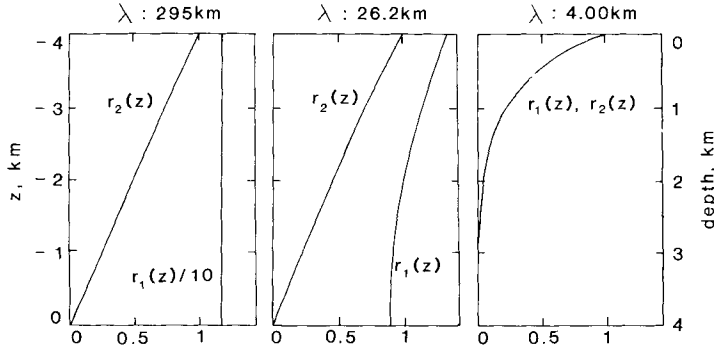


Figure 3. Tsunami displacement eigenfunctions in the ocean, normalized so that $r_2 \approx 1$ at the ocean surface. The model parameters are as in Fig. 2.

in the limit $\mu \rightarrow \infty$. Since $\rho c^2/\mu \ll 1$, the rigid bottom condition leads to a good approximation of the tsunami mode eigenfunctions, which is fine for representing tsunami propagation. However, although the quantities in (21) are very small with respect to the larger terms in (20), they cannot be neglected in modelling the excitation of the tsunami mode by sources within the solid earth; otherwise the excited waves would have zero amplitude!

Figs 3 and 4 illustrate examples of r_1 and r_2 calculated from (20) and (21), respectively, for three wavelengths typical of the three regimes of linear water waves (e.g. Le Méhauté 1976):

$\lambda = 295 \text{ km}$, $h/\lambda = 0.01$ — shallow water ($h/\lambda < 0.05$)

$\lambda = 26.2 \text{ km}$, $h/\lambda \approx 0.15$ — intermediate ($0.05 < h/\lambda < 0.5$)

$\lambda = 4.00 \text{ km}$, $h/\lambda = 1.0$ — deep water ($h/\lambda > 0.5$).

At the ocean surface r_1 and r_2 have the same sign at all three wavelengths, which corresponds to prograde particle motions (as can be seen from (1)). Fig. 3 shows that in the ocean horizontal displacements are much larger than vertical displacements for small h/λ . In the intermediate case the displacement components are comparable at the surface, while the horizontal component is much larger at depth, and for $h/\lambda > 0.5$ the components are almost equal at all depths (circular particle orbits) and both decrease rapidly with depth, becoming quite small near the ocean bottom. Although it does not show on the scale of Fig. 3, $r_2(z)$

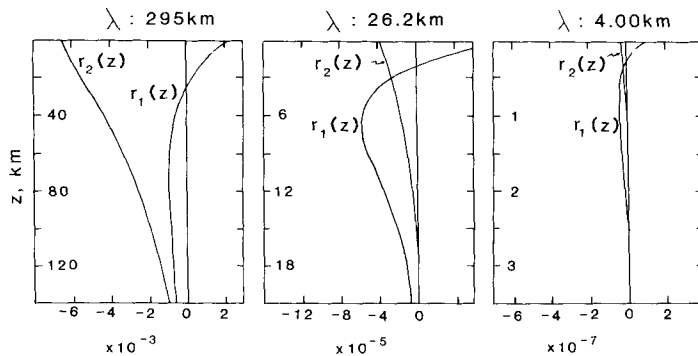


Figure 4. Tsunami displacement eigenfunctions in the solid earth with the same model parameters and normalization as Fig. 3.

does not vanish at $z = 0$ but is slightly negative there, changing sign when, approximately, $z = -3\rho c^2/(4\mu k)$. (This point is typically only a few metres above the ocean bottom.) Thus r_2 is indeed continuous at $z = 0$ (while r_1 is not), although a comparison of Figs 3 and 4 does not make this clear. As Fig. 4 shows, the displacement eigenfunctions are orders of magnitude smaller in the solid earth than near the ocean surface. Both components exhibit an eventual decay with depth, controlled by the factor $\exp(-kz)$ which appears in (21), and r_1 has a sign change near $z = 1/k$.

Finally, we note that the wavelengths used in Figs 3 and 4 were chosen deliberately to be comparable to the modes (of angular order 135, 1525 and 10 000) illustrated in comparable figures by Ward (1980). The two sets of results show essentially perfect agreement for the ocean layer, while the differences in the solid earth, which are quantitative rather than qualitative, are due to differences in the assumed elastic earth structure. The match is worst at the shortest wavelength, because only the top few kilometres of the solid earth are significantly penetrated by the displacement eigenfunctions and the elastic half-space is much more rigid in the upper few kilometres than the PEM-O model used by Ward.

Energy integrals and Hamilton's principle

In this section and the following two we turn to the problem of the excitation of the tsunami mode by point sources within the solid earth. We apply the variational technique of Saito (1967), extending its application to the Rayleigh-wave modes of a flat earth, as presented by Aki & Richards (1980), to the tsunami excitation problem.

Motivated by Hamilton's principle, we construct a Lagrangian L for the tsunami mode in which the energy densities are averaged over time and horizontal spatial coordinates. We proceed separately in the fluid and solid domains, beginning with the solid. Following Aki & Richards (1980), we substitute (1) into the expression (AR7.63) for the Lagrangian density of an isotropic elastic solid, average so that the terms $\sin^2(kx - \omega t)$ and $\cos^2(kx - \omega t)$ are replaced by $1/2$, and integrate over z . Using brackets to denote the averaging process,

$$\langle L^{\text{solid}} \rangle = \frac{1}{2} [\omega^2 I_1 - k^2 I_2 - k I_3 - I_4] \quad (23)$$

where the energy integrals I_1, I_2, I_3 and I_4 are given by

$$\begin{aligned} I_1 &= \frac{1}{2} \int_0^\infty \rho_s (r_1^2 + r_2^2) dz \\ I_2 &= \frac{1}{2} \int_0^\infty [(\lambda + 2\mu) r_1^2 + \mu r_2^2] dz \\ I_3 &= \int_0^\infty \left(\lambda r_1 \frac{dr_2}{dz} - \mu r_2 \frac{dr_1}{dz} \right) dz \\ I_4 &= \frac{1}{2} \int_0^\infty \left[(\lambda + 2\mu) \left(\frac{dr_2}{dz} \right)^2 + \mu \left(\frac{dr_1}{dz} \right)^2 \right] dz \end{aligned} \quad (24)$$

as in (AR7.74). I_1 pertains to the kinetic energy in the solid and I_2, I_3 and I_4 pertain to the elastic potential energy.

In the fluid there are two terms contributing to the Lagrangian, corresponding to the kinetic energy and the gravitational potential energy. The kinetic energy contribution can be expressed in terms of an integral analogous to I_1 , and it can in fact be integrated using Gauss' divergence theorem, yielding an expression containing boundary terms. The potential energy

contribution inherently involves boundary terms, since the tsunami is primarily a gravity wave on the ocean's free surface.

The kinetic energy density in the fluid is

$$\frac{1}{2} \rho \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 \right]. \quad (25)$$

Substituting for u and w from (1), averaging over t and x as before, and integrating from $-h$ to 0 over z we find

$$\langle \text{kinetic energy} \rangle = \frac{1}{2} \omega^2 I_5 \quad (26)$$

where

$$I_5 = \frac{1}{2} \rho \int_{-h}^0 (r_1^2 + r_2^2) dz. \quad (27)$$

One can also express the kinetic energy within some volume of an ideal, incompressible fluid in irrotational motion as an integral over the surface A of the volume, through the use of the divergence theorem. Following Fetter & Walecka (1980, equation 54.45), within the volume

$$\text{kinetic energy} = \frac{1}{2} \rho \int_A \phi \nabla \phi \cdot dA \quad (28)$$

where dA is directed outward from the volume and ϕ is the velocity potential, given by (12) for our purposes. If a portion of the surface bounding volume coincides with the free surface of the ocean $z = -h$, then on that portion $dA = -\hat{z}dA$. Similarly, for a portion coinciding with the ocean bottom $z = 0$, $dA = \hat{z}dA$. Hence by choosing a prismatic volume bounded by the free surface and ocean bottom, then averaging over time and horizontal coordinates, we find

$$\langle \text{kinetic energy} \rangle = \frac{1}{2} \rho \left\langle \phi \frac{\partial \phi}{\partial z} \right|_{z=-h}^{z=0} \rangle. \quad (29)$$

Using (12) and (15) to evaluate ϕ and $\partial \phi / \partial z$, (29) becomes identical to (26), $\langle \text{kinetic energy} \rangle = \frac{1}{2} \omega^2 I_5$, except that I_5 is given by

$$I_5 = \frac{1}{2} \frac{\rho}{k} [r_1(-h) r_2(-h) - r_1^{\text{fluid}}(0) r_2(0)]. \quad (30)$$

(27) and (30) must, of course, be equivalent. A way to confirm this equivalence is to integrate (27) by parts using (15). Both expressions for I_5 are useful.

Again following Fetter & Walecka (1980, equation 54.51), the gravitational potential energy of the fluid is given by

$$\text{gravitational potential energy} = \frac{1}{2} \rho g \int_A \eta^2 dA \quad (31)$$

where η is given by (6) and (1), and A is an area on the free surface $z = -h$ of the fluid. A term involving the deformation of the lower surface of the fluid is neglected in (31), but this is consistent with our neglect of gravity in the solid where a term of the same order is ignored, and η is in practice much larger than the amplitudes of the vertical displacement of

the ocean bottom. Next, averaging as before, we find

$$\langle \text{gravitational potential energy} \rangle = \frac{1}{2} I_6 \quad (32)$$

where

$$I_6 = \frac{1}{2} \rho g [r_2(-h)]^2. \quad (33)$$

Finally, subtracting (32) from (26), we have the Lagrangian in the fluid

$$\langle L^{\text{fluid}} \rangle = \frac{1}{2} [\omega^2 I_5 - I_6], \quad (34)$$

and combining (23) with (34) gives the (averaged) Lagrangian for the ocean–earth system,

$$\langle L \rangle = \langle L^{\text{solid}} \rangle + \langle L^{\text{fluid}} \rangle = \frac{1}{2} [\omega^2 I_1 - k^2 I_2 - k I_3 - I_4 + \omega^2 I_5 - I_6]. \quad (35)$$

The next step is to confirm Hamilton's principle, that is, to demonstrate that $\langle L \rangle$ is stationary for displacement-stress vectors satisfying (3) and (15), and boundary conditions (16), (17) and (18). We first show that $\langle L \rangle$ itself vanishes in this case.

We begin by multiplying the third equation of (3) by r_1 , the fourth equation of (3) by r_2 , and adding. Next we integrate from $z = 0$ to ∞ , by parts where possible, and finally use the first two equations of (3) to eliminate r_3 and r_4 from the integrands in which they remain. This yields

$$\omega^2 I_1 - k^2 I_2 - k I_3 - I_4 = -\frac{1}{2} [r_1 r_3 + r_2 r_4]_{z=0}^{\infty}. \quad (36)$$

(30) and (32) give the analogous result for the fluid,

$$\omega^2 I_5 - I_6 = \frac{1}{2} \left(\frac{\omega^2 \rho}{k} [r_1(-h)r_2(-h) - r_1(0)r_2(0)] - \rho g [r_2(-h)]^2 \right). \quad (37)$$

Adding (36) and (37), imposing the boundary conditions (16), (17) and (18), and noting that in (37), $r_1(0) = r_1^{\text{fluid}}(0)$, we find

$$\omega^2 I_1 - k^2 I_2 - k I_3 - I_4 + \omega^2 I_5 - I_6 = 0. \quad (38)$$

Hence by (35) the averaged Lagrangian vanishes.

We turn now to the task of demonstrating that $\langle L \rangle$ is stationary with respect to variations in $r_1(z)$ and $r_2(z)$, as suggested by Hamilton's principle. That is, we show that $\delta \langle L \rangle$ vanishes to first order in the variations $\delta r_1(z)$ and $\delta r_2(z)$.

From (24) it follows that, for the solid,

$$\omega^2 \delta I_1 - k^2 \delta I_2 - k \delta I_3 - \delta I_4 = [-r_3 \delta r_1 - r_4 \delta r_2]_0^{\infty} \quad (39)$$

using the relations $\delta(dr_1/dz) = d(\delta r_1)/dz$ and $\delta(dr_2/dz) = d(\delta r_2)/dz$, applying integration by parts to the terms containing these quantities, and simplifying some terms using (3).

For the fluid it is straightforward to show from (33) that

$$\delta I_6 = \rho g r_2(-h) \delta r_2(-h). \quad (40)$$

A similar expression for δI_5 follows directly from (30), but it is not useful here because it involves $\delta r_1(0)$, which has different values in solid and fluid. (This ambiguity does not occur in (39) because there $\delta r_1(0)$ is multiplied by $r_3(0)$, which always vanishes.) Instead, we start

from (27), which yields

$$\begin{aligned}
 \delta I_5 &= \rho \int_{-h}^0 (r_1 \delta r_1 + r_2 \delta r_2) dz \\
 &= \rho \int_{-h}^0 \left(-\frac{1}{k} r_1 \frac{d}{dz} \delta r_2 + r_2 \delta r_2 \right) dz \\
 &= -\frac{\rho}{k} r_1 \delta r_2 \Big|_{-h}^0 + \rho \int_{-h}^0 \left(\frac{1}{k} \frac{dr_1}{dz} + r_2 \right) \delta r_2 dz \\
 &= -\frac{\rho}{k} r_1 \delta r_2 \Big|_{-h}^0
 \end{aligned} \tag{41}$$

using $\delta r_1 = (-1/k)d(\delta r_2)/dz$ and $(1/k)dr_1/dz + r_2 = 0$, which both follow from (15). This has the desired property of not involving $\delta r_1(0)$.

Finally, it follows from (35), (39), (40) and (41) that

$$\begin{aligned}
 2\delta \langle L \rangle &= \omega^2 \delta I_1 - k^2 \delta I_2 - k \delta I_3 - \delta I_4 + \omega^2 \delta I_5 - \delta I_6 = -r_3(\infty) \delta r_1(\infty) - r_4(\infty) \delta r_2(\infty) \\
 &\quad + r_3(0) \delta r_1^{\text{solid}}(0) + r_4(0) \delta r_2(0) - \rho \frac{\omega^2}{k} r_1^{\text{fluid}}(0) \delta r_2(0) \\
 &\quad + \rho \frac{\omega^2}{k} r_1(-h) \delta r_2(-h) - \rho g r_2(-h) \delta r_2(-h).
 \end{aligned} \tag{42}$$

$r_3(\infty) = r_4(\infty) = 0$, from (3) and (18), and imposing the boundary conditions (16) and (17), the right-hand side of (42) vanishes. Thus, in accord with Hamilton's principle,

$$\omega^2 \delta I_1 - k^2 \delta I_2 - k \delta I_3 - \delta I_4 + \omega^2 \delta I_5 - \delta I_6 = 0. \tag{43}$$

A useful formula for tsunami group velocity follows from the results that the Lagrangian (35) vanishes when $\mathbf{r}(\mathbf{z})$ satisfies the equations of motion and boundary conditions corresponding to a tsunami, and that it is stationary about that solution. By varying ω and k in (38) and applying (43) one finds $2\omega \delta \omega I_1 - 2k \delta k I_2 - \delta k I_3 + 2\omega \delta \omega I_5 = 0$, and equating U to $\delta \omega / \delta k$,

$$U = \frac{I_2 + I_3/2k}{c(I_1 + I_5)} \tag{44}$$

where $c = \omega/k$ is the phase velocity.

Point force response: the tsunami residue

Here we consider the response of the ocean–earth system to a point force (F_x, F_y, F_z) applied at $x = 0, y = 0, z = d$ with $d > 0$ and varying harmonically in time according to $\exp(-i\omega t)$. It will be shown that the tsunami normal mode eigenfunctions provide a compact representation of the response in the far-field.

As Aki & Richards (1980) state, the point force is equivalent to a discontinuity in traction on the horizontal plane $z = d$ such that

$$\begin{aligned}
 T(d^+) - T(d^-) &= -F \exp(-i\omega t) \delta(x) \delta(y) = \frac{\exp(-i\omega t)}{2\pi} \sum_m \int_0^\infty k [f_T(k, m) T_k^m(r, \phi) \\
 &\quad + f_S(k, m) S_k^m(r, \phi) + f_R(k, m) R_k^m(r, \phi)] dk
 \end{aligned} \tag{45}$$

(AR7.119, AR7.120) where T_k^m , S_k^m and R_k^m are the cylindrical vector harmonics defined in (AR7.117) and (AR7.113). We now use a cylindrical coordinate system (r, ϕ, z) such that $x = r \cos \phi$ and $y = r \sin \phi$ with orthogonal unit vectors $\hat{r}$, $\hat{\phi}$ and $\hat{z}$. The coefficients f_T , f_S and f_R , are given by (AR7.125), (AR7.127) and (AR7.128), respectively.

The solution for the displacement, in solid and liquid, due to the time-harmonic point force has the form given by (AR7.131),

$$u(r, \phi, z, t) = \frac{\exp(-i\omega t)}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} k [l_1(k, m, z, \omega) T_k^m(r, \phi) + r_1(k, m, z, \omega) S_k^m(r, \phi) + r_2(k, m, z, \omega) R_k^m(r, \phi)] dk \quad (46)$$

and the corresponding horizontal traction is

$$T(r, \phi, z, t) = \frac{\exp(-i\omega t)}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} k [l_2(k, m, z, \omega) T_k^m(r, \phi) + r_3(k, m, z, \omega) S_k^m(r, \phi) + r_4(k, m, z, \omega) R_k^m(r, \phi)] dk \quad (47)$$

where it is important to note that $\mathbf{r} = (r_1, r_2, r_3, r_4)$ and $\mathbf{l} = (l_1, l_2)$ do not represent normal mode eigenfunctions, because the equations of motion now include the point force, and hence are no longer homogeneous, so that there is no longer a dispersion relation and k is independent of ω .

It is easy to show, however, that except at the source depth the equations of motion in the solid are satisfied if $\mathbf{r}$ satisfies (3) and $\mathbf{l}$ satisfies (AR7.24), which were derived for the Rayleigh-wave (and tsunami) and Love-wave displacement-stress normal mode eigenfunctions, respectively. In the fluid, where shear stresses must vanish, $\mathbf{l} = 0$ and the equations governing $\mathbf{r}$ can be readily demonstrated by introducing the velocity potential

$$\Phi(r, \phi, z, t) = i\omega \frac{\exp(-i\omega t)}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} r_1(k, m, z, \omega) Y_k^m(r, \phi) dk \quad (48)$$

where $Y_k^m(r, \phi)$ is the scalar harmonic defined by (AR7.113). Requiring that (46) and (48) be consistent with (4) and (5) implies that r_1 and r_2 must satisfy (15), and by requiring that shear tractions vanish in the fluid and that (47) and (48) be consistent with (9), we find that r_3 and r_4 must satisfy (14). Finally, substituting (48) into (11) and invoking (15), it can be shown that r_1 and r_2 satisfy (17).

In addition to the free surface condition, $\mathbf{r}$ and $\mathbf{l}$ must satisfy boundary conditions ensuring the continuity of the vertical displacement and vertical traction at the solid–fluid interface, yielding (16) with k -, m - and ω -dependence implicit, and $l_2(k, m, 0, \omega) = 0$. Also, for $z \rightarrow \infty$ $\mathbf{l}$ and $\mathbf{r}$ must either vanish or correspond to down-going body waves, and last, but very important, is the condition that at $z = d$ the traction have the discontinuity prescribed by (45). Thus

$$\begin{aligned} l_2(k, m, d+, \omega) - l_2(k, m, d-, \omega) &= f_T(k, m) \\ r_3(k, m, d+, \omega) - r_3(k, m, d-, \omega) &= f_S(k, m) \\ r_4(k, m, d+, \omega) - r_4(k, m, d-, \omega) &= f_R(k, m). \end{aligned} \quad (49)$$

We next obtain the tsunami component of (46), which clearly depends on $\mathbf{r}$ and not on $\mathbf{l}$. $\mathbf{r}$ can be constructed in the following manner,

$$\mathbf{r} = \mathbf{r}' + \frac{\mathbf{r}''}{\Delta(k, m, \omega)}. \quad (50)$$

We require $\mathbf{r}'$ and $\mathbf{r}''$ each satisfy (3) in the solid and (14) and (15) in the fluid, and that they independently satisfy the interface boundary conditions of continuity of vertical displacement and continuity of the vertical component of traction on the horizontal plane. Hence

$$r_2' \text{ solid}(0) = r_2' \text{ fluid}(0) \quad (51a)$$

$$r_2'' \text{ solid}(0) = r_2'' \text{ fluid}(0) \quad (51b)$$

$$r_4' \text{ solid}(0) = r_4' \text{ fluid}(0) = \frac{\rho\omega^2}{k} r_1' \text{ fluid}(0) \quad (51c)$$

$$r_4'' \text{ solid}(0) = r_4'' \text{ fluid}(0) = \frac{\rho\omega^2}{k} r_1'' \text{ fluid}(0), \quad (51d)$$

where, for convenience, we show explicitly only the z -dependence of $\mathbf{r}'$ and $\mathbf{r}''$. In addition, we let $\mathbf{r}''$ be continuous at $z = d$ and require that it satisfy the radiation condition (down-going waves or vanishing displacements) while $\mathbf{r}' = 0$ for $z > d$ and $\mathbf{r}'$ has a discontinuity in stress at $z = d$, so that

$$\begin{aligned} r_1'(d-) &= 0 \\ r_2'(d-) &= 0 \\ r_3'(d-) &= -f_S \\ r_4'(d-) &= -f_R. \end{aligned} \quad (52)$$

The boundary conditions (51a), (51c) and (52) completely constrain $\mathbf{r}'$. Also, given the definitions of and conditions on $\mathbf{r}'$ and $\mathbf{r}''$, it is clear that $\mathbf{r}$ as given by (50) satisfies the equations and boundary conditions governing it, except for (17) and the vanishing of the shear traction at $z = 0$. These final two conditions yield

$$r_2''(-h) - \frac{\omega^2}{kg} r_1''(-h) + \Delta \left[r_2'(-h) - \frac{\omega^2}{kg} r_1'(-h) \right] = 0 \quad (53a)$$

$$r_3'' \text{ solid}(0) + \Delta r_3' \text{ solid}(0) = 0, \quad (53b)$$

competing the conditions on $\mathbf{r}''$ and also defining $\Delta(k, m, \omega)$.

The tsunami contribution to (46) comes from the ‘tsunami pole’ where ω and $k = k_t$ satisfy the tsunami dispersion relation and $\Delta(k_t, m, \omega) = 0$. Taking the residue at this pole we have

$$\begin{aligned} u^{\text{tsunami}}(r, \phi, z, t) = \exp(-i\omega t) \sum_{m=-\infty}^{\infty} ik_t \left[\frac{r_1''(k_t, m, z, \omega)}{(\partial\Delta/\partial k)_{k=k_t}} S_{k_t}^m(r, \phi) \right. \\ \left. + \frac{r_2''(k_t, m, z, \omega)}{(\partial\Delta/\partial k)_{k=k_t}} R_{k_t}^m(r, \phi) \right]. \end{aligned} \quad (54)$$

This is the same, essentially, as the corresponding result for Rayleigh waves, and the details of treating the integral over k are identical in both cases, except that for Rayleigh waves there may be several poles, corresponding to the fundamental mode and higher modes. Only one pole exists for the tsunami mode, equivalently there is only one branch of the tsunami dispersion curve; a more detailed discussion of this point has been given by Okal (1982). Note, however, that our model of the ocean is appropriate for tsunamis but not for Rayleigh waves, since the compressibility of the ocean is neglected. A similar model incorporating the compressibility of the ocean could be used for both tsunamis and Rayleigh waves, but has no advantage over the incompressible ocean model for the former. A further extension would be to include a realistic density stratification in the ocean, in which case the internal gravity modes of the ocean would be present (e.g. Lighthill 1978), but this complication would likewise have little advantage for the accurate modelling of tsunamis.

The displacement in (54) can be expressed in terms of the normal mode tsunami eigenfunctions and the energy integrals defined in the previous section. To show this we derive an expression for Δ and differentiate it with respect to k by means of a variational method. First we take note of two quantities having the useful property of being constant with respect to z . In the solid (and for $0 \leq z < d$ in particular),

$$\frac{d}{dz} (r_1'' r_3' - r_1' r_3'' + r_2'' r_4' - r_2' r_4'') = 0 \quad (55)$$

which follows directly from the fact that $\mathbf{r}'$ and $\mathbf{r}''$ both satisfy (3), hence $r_1'' r_3' - r_1' r_3'' + r_2'' r_4' - r_2' r_4''$ is invariant with respect to z in the solid. Similarly, in the fluid ($-h \leq z \leq 0$) we have from (15)

$$\frac{d}{dz} (r_1' r_2'' - r_2' r_1'') = 0 \quad (56)$$

so that $r_1' r_2'' - r_2' r_1''$ is invariant.

Now, combining (51c) with (51d) we have $r_4''(0) - \rho(\omega^2/k) r_1''^{\text{fluid}}(0) + \Delta[r_4'(0) - \rho(\omega^2/k) r_1'^{\text{fluid}}(0)] = 0$. Multiplying this result by $r_2''(0)$, multiplying (53a) by $-\rho g r_2''(-h)$, multiplying (53b) by $r_1''^{\text{solid}}(-h)$, adding, and solving for Δ yields

$$\Delta = -\gamma_1/\gamma_2 \quad (57)$$

where

$$\gamma_1 = r_1''^{\text{solid}}(0) r_3''^{\text{solid}}(0) + r_2''(0) r_4''(0) + \rho \frac{\omega^2}{k} [r_1''(-h) r_2''(-h) - r_1''^{\text{fluid}}(0) r_2''(0)] - \rho g [r_2''(-h)]^2 \quad (58a)$$

and

$$\gamma_2 = r_1''^{\text{solid}}(0) r_3'^{\text{solid}}(0) + r_2''(0) r_4'(0) - \rho \frac{\omega^2}{k} r_1'^{\text{fluid}}(0) r_2''(0) + \rho \frac{\omega^2}{k} r_1'(-h) r_2''(-h) - \rho g r_2'(-h) r_2''(-h). \quad (58b)$$

Defining $I_1'', I_2'', I_3'', I_4'', I_5''$ and I_6'' by (24), (30) and (33) with $\mathbf{r}''$ substituted for $\mathbf{r}$, it is clear from (35), (36) and (37) that (58a) can be rewritten as

$$\gamma_1 = 2[\omega^2 I_1'' - k^2 I_1'' - k I_3'' - I_4'' + \omega^2 I_5'' - I_6''] + r_1''(\infty) r_3''(\infty) + r_2''(\infty) r_4''(\infty). \quad (59)$$

Next we consider the variation in Δ due to a variation in k about the point where $k = k_t$ and $\Delta(k, m, \omega) = 0$ or, equivalently, where $\mathbf{r}''$ is a tsunami eigenfunction. At this point $\gamma_1 = 0$ according to (59), (38) and (18) with $\mathbf{r}''$ substituted for $\mathbf{r}$. Hence the variation $\delta\gamma_2$ has no effect on $\delta\Delta$ and we have simply

$$\delta\Delta = -\delta\gamma_1/\gamma_2. \quad (60)$$

From (59) we obtain, using (43), (18) and the requirement that $\delta r_1''(\infty) = \delta r_2''(\infty) = 0$,

$$\delta\gamma_1 = -(4kI_2'' + 2I_3'')\delta k. \quad (61)$$

The evaluation of γ_2 at the point where $\mathbf{r}''$ is a tsunami eigenfunction is accomplished with the aid of the two depth-independent quantities noted previously. From (17), (56) and (51d)

$$\begin{aligned} \rho \frac{\omega^2}{k} r_1''(-h) r_2''(-h) - \rho g r_2'(-h) r_2''(-h) \\ = \rho \frac{\omega^2}{k} [r_1'(-h) r_2''(-h) - r_2'(-h) r_1''(-h)] \\ = \rho \frac{\omega^2}{k} [r_1'^{\text{fluid}}(0) r_2''(0) - r_2'(0) r_1''^{\text{fluid}}(0)] \\ = \rho \frac{\omega^2}{k} r_1'^{\text{fluid}}(0) r_2''(0) - r_2'(0) r_4''(0). \end{aligned} \quad (62)$$

Then, combining (62) and (58b) and applying (16b), (55) and (52) we find

$$\begin{aligned} \gamma_2 = r_1''^{\text{solid}}(0) r_3'^{\text{solid}}(0) - r_1'^{\text{solid}}(0) r_3''^{\text{solid}}(0) + r_2''(0) r_4'(0) - r_2'(0) r_4''(0) \\ = r_1''(d) r_3'(d-) - r_1'(d-) r_3''(d) + r_2''(d) r_4'(d-) - r_2'(d-) r_4''(d) \\ = -f_S r_1''(d) - f_R r_2''(d). \end{aligned} \quad (63)$$

Finally, substituting (61) and (63) into (60), it is clear that

$$\left(\frac{\partial\Delta}{\partial k}\right)_{k=k_t} = -\frac{4k_t I_2'' + 2I_3''}{f_S r_1''(d) + f_R r_2''(d)} \quad (64)$$

where the quantities on the right are all evaluated at $k = k_t$. Using (44), an equivalent relation is

$$\frac{1}{(\partial\Delta/\partial k)_{k=k_t}} = -\frac{f_S(k_t, m) r_1''(d) + f_R(k_t, m) r_2''(d)}{4k_t cU(I_1'' + I_5'')} \quad (65)$$

The result (65) may be readily substituted in (54). The sum over m is readily evaluated using the values assigned to f_S and f_R in (AR7.127) and (AR7.128), $f_S(k, 1) = (-F_x + iF_y)/2$, $f_S(k, -1) = (F_x + iF_y)/2$, and $f_S = 0$ for $m \neq \pm 1$, and $f_R(k, 0) = F_z$ and $f_R = 0$ for $m \neq 0$. Since at the pole where $k = k_t$, $\mathbf{r}''(k_t, m, z, \omega)$ is a tsunami eigenfunction, we drop the primes and replace it by $\mathbf{r}(z)$, reverting to our earlier notation. Likewise we replace k_t by k , I_1'' by I_1 , and I_5'' by I_5 , but note that these quantities are all implicit functions of ω . Also, we are interested only in the result for the far-field, where the tsunami is distinct from other waves radiated by the point force. Thus we replace $S_k^m(r, \phi)$ and $R_k^m(r, \phi)$ by the leading terms of their large r asymptotic expansions corresponding to outgoing waves, given by

(AR7.118). Finally we have the far-field tsunami generated by a time-harmonic point force at depth d in the solid earth:

$$u^{\text{tsunami}}(r, \phi, z, t) = \frac{\exp(-i\omega t)}{8cU(I_1 + I_5)} [F_z r_2(d) + i(F_x \cos \phi + F_y \sin \phi) r_1(d)] \\ \times \left(\frac{2}{\pi k r} \right)^{1/2} [r_1(z) \exp(-i\pi/4) \hat{r} + r_2(z) \exp(i\pi/4) \hat{z}] \exp(ikr). \quad (66)$$

This is identical to the corresponding result for Rayleigh waves (AR7.143) except that there is no sum over higher modes, the kinetic energy for the ocean I_5 has been added to the kinetic energy integral I_1 for the solid earth and, of course, r_1 and r_2 denote the tsunami normal mode displacement eigenfunctions. Hence (66) is not surprising, but neither is it trivial.

Tsunami from a point moment tensor source

The far-field tsunami response to a point moment tensor source in the solid earth can be obtained by differentiating the second rank tensor Green's function implicitly stated by (66). This procedure parallels perfectly that for Rayleigh waves, and just as the Rayleigh wave response to a point moment tensor source (AR7.149 and AR7.150), follows from (AR7.143) we have from (66)

$$u_z^{\text{tsunami}}(r, \phi, z, \omega) = \frac{r_2(z)}{8cU(I_1 + I_5)} \left(\frac{2}{\pi k r} \right)^{1/2} \exp[i(kr + \pi/4)] \\ \times \left\{ [M_{xx} \cos^2 \phi + 2M_{xy} \sin \phi \cos \phi + M_{yy} \sin^2 \phi] kr_1(d) \right. \\ \left. + M_{zz} \frac{dr_2}{dz}(d) + i[M_{xz} \cos \phi + M_{yz} \sin \phi] \left(\frac{dr_1}{dz}(d) - kr_2(d) \right) \right\} \quad (67a)$$

$$u_r^{\text{tsunami}}(r, \phi, z, \omega) = \frac{r_1(z)}{8cU(I_1 + I_5)} \left(\frac{2}{\pi k r} \right)^{1/2} \exp[i(kr - \pi/4)] \{ \} \quad (67b)$$

where the quantity in braces $\{ \}$ in (67b) is the same as that in (67a). Note that these equations are in the frequency domain; each moment tensor component M_{ij} is a function of ω given by the Fourier transform of its time domain representation $M_{ij}(t)$. Also note that the moment tensor must be symmetric, since it represents an internal source that applies no torque.

From (67) both the vertical and radial components of the tsunami from a point moment-tensor source can be synthesized at any depth in the ocean or earth. Since the vertical displacement at the ocean surface is usually the quantity of greatest interest, we note from (6) with $u_z = w$, and (67a) that

$$\eta(r, \phi, \omega) = \frac{-r_2(-h)S(\omega)}{8cU(I_1 + I_5)} \left(\frac{2}{\pi k r} \right)^{1/2} \exp[i(kr + \pi/4)] \{ \} \quad (68)$$

As before η is positive for upward displacement of the ocean surface, that is, displacement in the negative z -direction. Here we have made the additional simplification of assuming that

all the moment tensor components share a common time dependence, so that $M_{ij}(t) = S(t)M_{ij}$ where $M_{ij} = M_{ij}(\infty)$ is now the static moment tensor, $S(t = \infty) = 1$, and

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(t) \exp(i\omega t) dt \quad (69)$$

is the Fourier transform of the source time history. The time domain waveform of the tsunami due to the point moment tensor source can thus be obtained by operating on (68) with the inverse Fourier transform

$$\int_{-\infty}^{\infty} \exp(-i\omega t) d\omega.$$

The relation (68) can thus be expressed in the time domain as

$$\begin{aligned} \eta(r, \phi, t) = & [M_{xx} \cos^2 \phi + 2M_{xy} \sin \phi \cos \phi + M_{yy} \sin^2 \phi] f_1(r, t) + M_{zz} f_2(r, t) \\ & + [M_{xz} \cos \phi + M_{yz} \sin \phi] f_3(r, t) \end{aligned} \quad (70)$$

where $f_1(r, t)$, $f_2(r, t)$ and $f_3(r, t)$ are the inverse transforms, respectively, of

$$\begin{aligned} f_1(r, \omega) &= -r_2(-h)N(\omega) S(\omega) (h/r)^{1/2} \exp[i(kr + \pi/4)] kr_1(d) \\ f_2(r, \omega) &= -r_2(-h)N(\omega) S(\omega) (h/r)^{1/2} \exp[i(kr + \pi/4)] \frac{dr_2}{dz}(d) \\ f_3(r, \omega) &= -r_2(-h)N(\omega) S(\omega) (h/r)^{1/2} \exp[i(kr + 3\pi/4)] \left(\frac{dr_1}{dz}(d) - kr_2(d) \right) \end{aligned} \quad (71)$$

where

$$N(\omega) = \frac{1}{8cU(I_1 + I_5)} \left(\frac{2}{\pi k h} \right)^{1/2}. \quad (72)$$

This remarkably simple result permits many relations to be readily examined. For example, from (71) and the approximate eigenfunctions (21) we find that for the uniform elastic solid earth the relative tsunami excitation versus depth is governed by

$$\begin{aligned} |f_1| &\propto |kz - 1/2| \exp(-kz) \\ |f_2| &\propto (kz + 1/2) \exp(-kz) \\ |f_3| &\propto 2kz \exp(-kz). \end{aligned} \quad (73)$$

The expressions on the right-hand side of (73) are plotted in Fig. 5 from which it is clear that except at extremely shallow depths the moment tensor components M_{zz} , M_{xz} and M_{yz} have the largest contributions to the tsunami amplitude. Also evident are the eventual decay of tsunami excitation with depth and the fact that this decay occurs more rapidly for short wavelengths and, conversely, that shallow sources produce more short-wavelength radiation than deep sources. Fixing the source depth d we can then plot the complex spectra given by (71) and the corresponding waveforms in the time domain, but first it is necessary to choose the function $S(t)$.

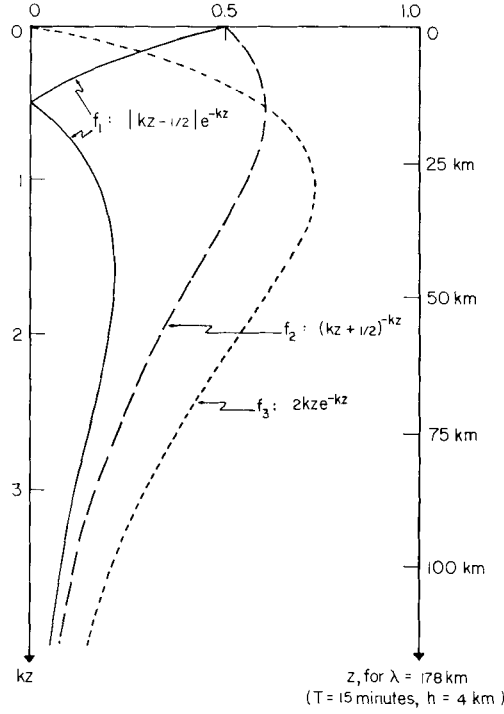


Figure 5. Relative tsunami excitation versus depth for an ocean overlying an elastic half-space.

We consider two choices, $S(t) = H(t)$ and $S(t) = R(\tau, t)$. Here $H(t)$ denotes the step function,

$$H(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ 1 & t > 0 \end{cases} \quad (74a)$$

and $R(\tau, t)$ denotes the ramp function with rise time τ ,

$$R(\tau, t) = \begin{cases} 0 & t \leq 0 \\ t/\tau & 0 \leq t \leq \tau \\ 1 & t \geq \tau \end{cases} \quad (74b)$$

Of course, these coincide in the limit $\tau \rightarrow 0$. Taking the Fourier transforms of (74a) and (74b), following (69), we find

$$H(\omega) = \pi\delta(\omega) + i/\omega \quad (75a)$$

$$\begin{aligned} R(\tau, \omega) &= \pi\delta(\omega) + \frac{\exp(i\omega\tau) - 1}{\omega^2\tau} \\ &= \pi\delta(\omega) + \frac{2 \sin(\omega\tau/2)}{\omega^2\tau} \exp[i(\pi + \omega\tau)/2]. \end{aligned} \quad (75b)$$

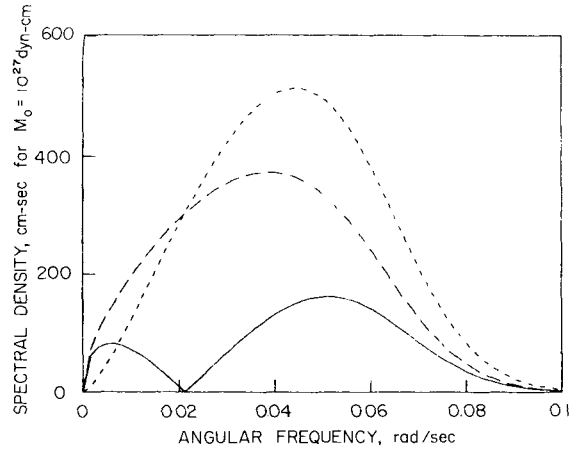


Figure 6. Amplitude spectral of f_1 (solid line), f_2 (dashed line) and f_3 (dotted line), for $r = 3000$ km, $d = 5$ km, $\tau = 0$, and the model parameters given in the text.

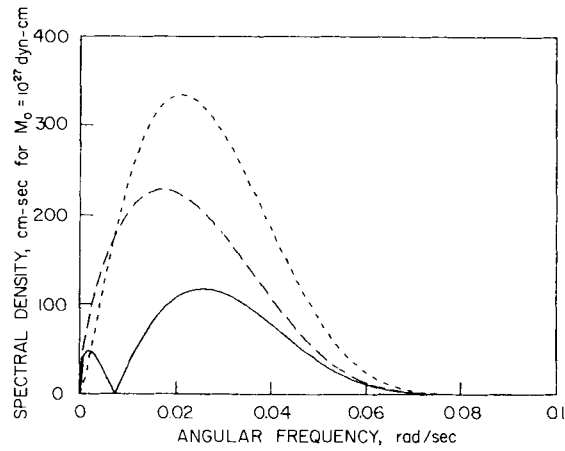


Figure 7. Amplitude spectra of f_1 , f_2 and f_3 as in Fig. 6 but with $d = 15$ km.

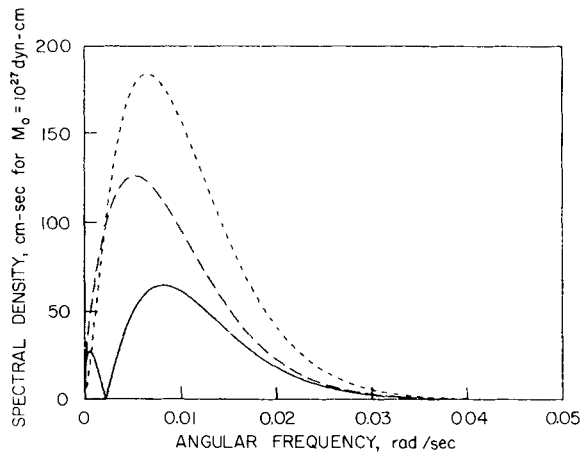


Figure 8. Amplitude spectra of f_1 , f_2 and f_3 as in Fig. 6 but with $d = 50$ km.

We now present several examples of f_1 , f_2 and f_3 and examine their form in both the time and frequency domains. Details on the numerical computations are given by Comer (1982). Here we simply note that the waveform synthesis involved an application of the Fast Fourier Transform algorithm via the FORTRAN subroutine FOUR1 by Norman Brenner (MIT Lincoln Laboratory 1967) and that the number of computations was reduced by interpolating in frequency using the FORTRAN subroutine SPLINE by Forsythe, Malcolm & Moler (1977). In all the examples the ocean depth is set at 5 km and that the solid earth is an elastic half-space with $\alpha = 7.1 \text{ km s}^{-1}$, $\beta = 4.1 \text{ km s}^{-1}$ and $\rho_s = 3.1 \text{ g cm}^{-3}$. Initially we take $S(t) = H(t)$ and $r = 3000 \text{ km}$.

Figs 6, 7 and 8 show the amplitude spectra of f_2 , f_1 and f_3 for different source depths, $d = 5, 15$ and 50 km , respectively. The spectra are quite smooth, which is not surprising, except that in all three figures f_1 has a zero. This reflects the zero in its excitation curve as shown in Fig. 5. A decrease in amplitude with increasing source depth is evident, especially for higher frequencies. It is also clear that the amplitude spectrum of f_1 is substantially smaller than the other two. The fact that we have chosen a particular value of r for these examples is unimportant, since it is clear from (71) that the amplitude spectra scale according to $r^{-1/2}$ independently of ω .

Fig. 9 illustrates the phase spectra of f_1 and f_2 , which are identical if one neglects the jump of π in the phase of f_1 at the point where $|f_1(\omega)| = 0$. The phase spectrum of f_3 is identical to the others except for a phase advance of $\pi/2$, which is indistinguishable on the scale of the figure. We must note that the phase illustrated in Fig. 9 is not relative to the origin time of the event, but rather is relative to the long-wave arrival time $t_0 = r/\sqrt{gh}$. This corresponds to replacing $k(\omega)r$ by $k(\omega)r - \omega t_0$ in (71). The phase is close to zero for the low frequencies and a phase delay (reflecting the tsunami dispersion) occurs for the higher frequencies. As in the case of the amplitude spectra, the particular value of r is unimportant since, neglecting the frequency-independent terms, the phase spectra of f_1 , f_2 and f_3 are proportional to r only through the term $r[k(\omega) - \omega/\sqrt{gh}]$.

The waveforms synthesized from the spectra illustrated in Figs 6–9 are shown in Figs 10, 11 and 12 for source depths of 5, 15 and 50 km, respectively. As the amplitude spectra indicate, f_1 is much smaller than f_2 or f_3 . The zero in the spectrum of f_1 is reflected in the relative complexity of its waveform. Also, the overall amplitudes and the proportion of high frequencies decrease with source depth and the difference in the initial phase of f_3 compared

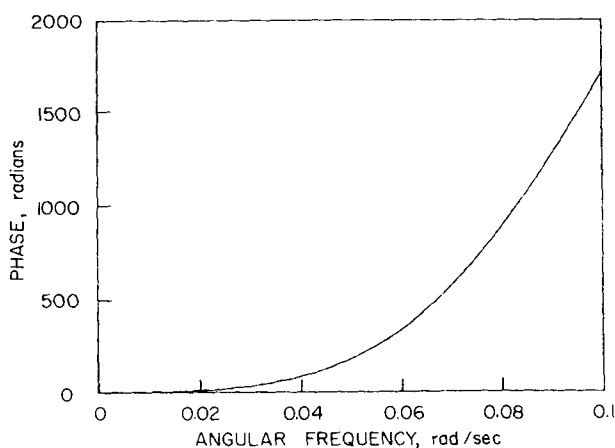


Figure 9. Phase spectrum, relative to $t_0 = r/(gh)^{1/2}$, of f_2 corresponding to Figs 6, 7 and 8. On this scale the phase spectra of f_1 and f_3 are indistinguishable from the curve shown here.

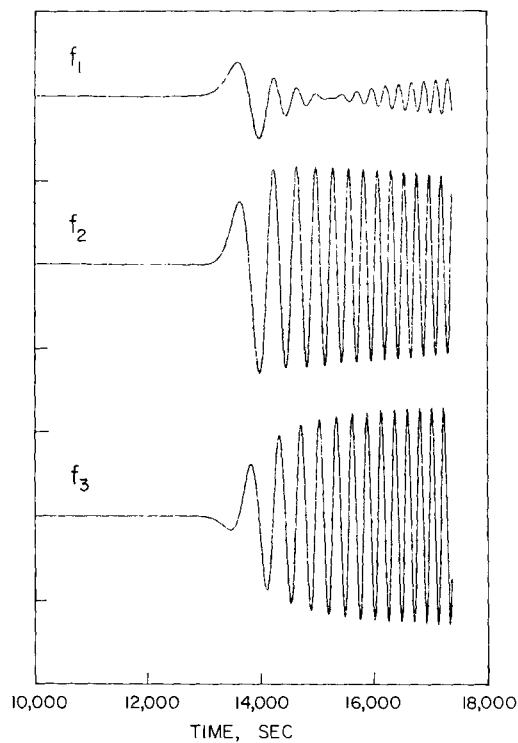


Figure 10. Elementary waveforms $f_1(r, t)$, $f_2(r, t)$ and $f_3(r, t)$ for $r = 3000 \text{ km}$, $d = 5 \text{ km}$, $\tau = 0$, and the model parameters given in the text. Each interval on the vertical scale is $0.5 \text{ cm}/10^{27} \text{ dyn cm}$.

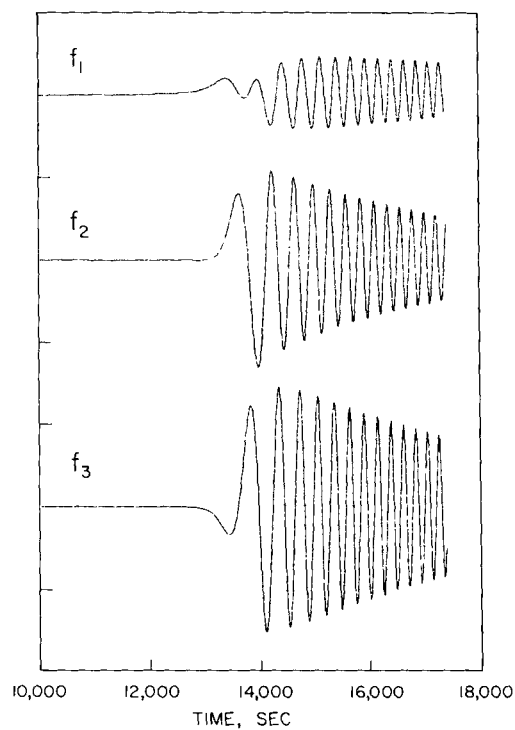


Figure 11. Elementary waveforms as in Fig. 10 for $r = 3000 \text{ km}$, $d = 15 \text{ km}$ and $\tau = 0$.

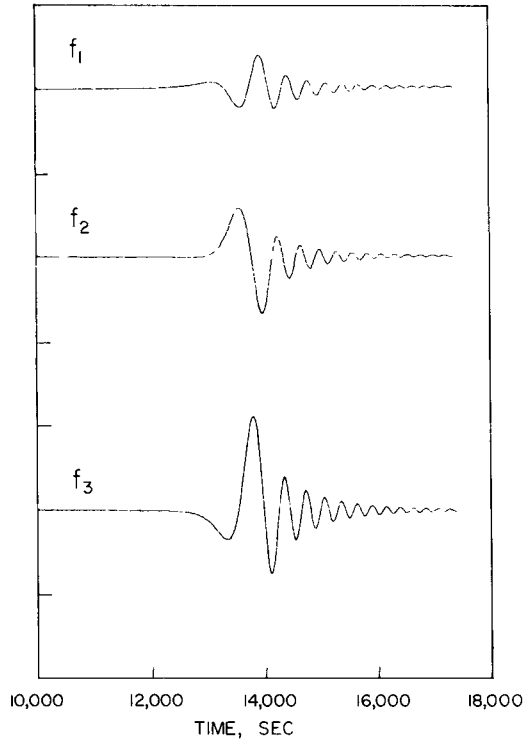


Figure 12. Elementary waveforms as in Fig. 10 for $r = 3000$ km, $d = 50$ km and $\tau = 0$.

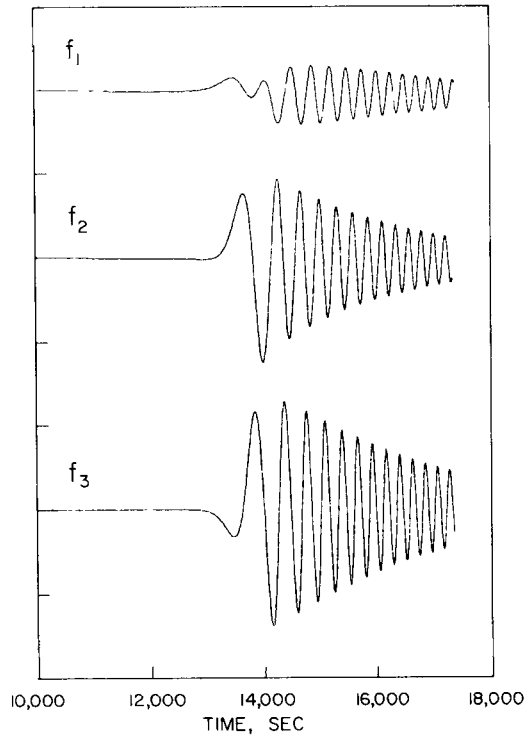


Figure 13. Elementary waveforms as in Fig. 10 for $r = 3000$ km, $d = 15$ km and $\tau = 120$ s.

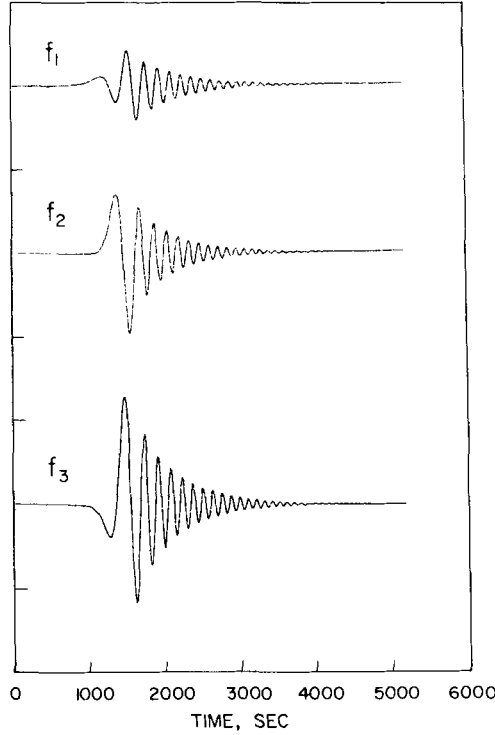


Figure 14. Elementary waveforms as in Fig. 10 for $r = 300$ km, $d = 15$ km and $\tau = 0$. Each interval on the vertical scale is $5 \text{ cm}/10^{27} \text{ dyn cm}$.

to f_1 and f_2 is evident. Like Fig. 11, Fig. 13 shows the waveforms for $r = 3000$ km and $d = 15$ km, but with $S(t) = R(\tau, t)$ and $\tau = 120$ s. Clearly setting the rise time even to this rather large value has little effect on the waveforms except for the later dispersive portions. Fig. 14 shows another example with $d = 15$ km and $\tau = 0$, but with $r = 300$ km. Closer to the source the waveforms are larger in amplitude and less dispersed, which is not surprising.

Having compared a range of examples of the elementary waveforms f_1 , f_2 and f_3 it is very instructive to see which ones are radiated by some simple double couple sources. For example, a source whose only non-zero moment tensor components are $M_{xy} = M_{yx} = M_0$ excites a tsunami of the form

$$\eta(r, \phi, t) = M_0 \sin 2\phi f_1(r, t). \quad (76a)$$

This is a pure strike-slip source whose lower-hemisphere focal mechanism is illustrated in Fig. 15(a). With non-zero components $M_{yy} = M_0$ and $M_{zz} = M_0$,

$$\eta(r, \phi, t) = M_0 [-\sin^2 \phi f_1(r, t) + f_2(r, t)] \quad (76b)$$

and the source has a pure thrust mechanism with a dip of 45° , as illustrated in Fig. 15(b). Finally, if $M_{yz} = M_{zy} = M_0$ are the only non-zero moment tensor components then

$$\eta(r, \phi, t) = M_0 \sin \phi f_3(r, t). \quad (76c)$$

This third example corresponds to the vertical dip-slip mechanism of Fig. 15(c). The arrows next to the mechanisms in Fig. 15 indicate azimuths for which the tsunamis have the simple forms $\eta = M_0 f_1$, $\eta = M_0 f_2$ and $\eta = M_0 f_3$. Note that since f_1 consistently has a much

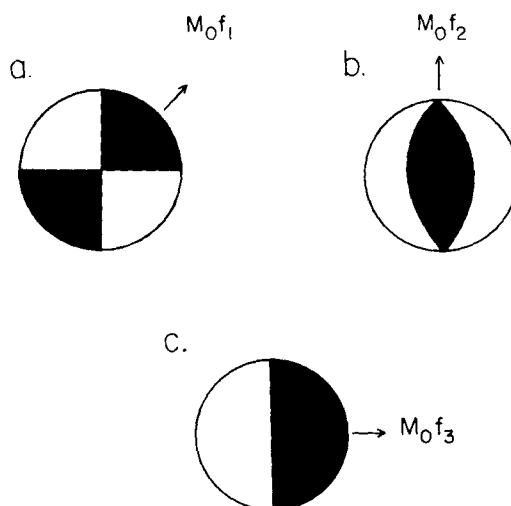


Figure 15. Three simple double couple focal mechanisms: (a) strike-slip, (b) 45° thrust and (c) vertical dip-slip. The lower hemisphere is shown and compressional quadrants are shaded. The arrows show directions in which the tsunami waveforms from a point source correspond to elementary waveforms multiplied by the scalar moment.

smaller amplitude than f_2 or f_3 in Figs 10–14, the strike-slip mechanism (Fig. 15a, equation 76a) is substantially less efficient for tsunami generation than the dip-slip mechanisms. This is in accord with the observation (e.g. Iida 1970) that tsunamis generated by strike-slip earthquakes are rare and never large, although the rarity of large submarine strike-slip earthquakes themselves is undoubtedly also a factor.

Finally, we note that the strike-slip mechanism of Fig. 15(a) and the vertical dip-slip mechanism of Fig. 15(c) correspond to examples presented by Ward (1980). Comer (1982) has shown by a direct comparison that the waveforms computed from the flat earth normal mode theory, for these two mechanisms, are in excellent agreement with Ward's. There are minor discrepancies, but these are surprisingly small in view of the substantial differences between the solid earth models that were used. Because of the efficiency of the Fast Fourier Transform (and, to a small extent, because of our use of analytic eigenfunctions) the flat earth calculations are substantially faster.

Discussion

We have solved the tsunami mode excitation problem for an arbitrary depth dependence of the density and the elastic moduli in the solid earth, but the computed examples are restricted to the use of the approximate, analytic eigenfunctions for an elastic half-space underlying the ocean. This restriction could be removed through a numerical solution of (3). As for the case of seismic surface waves, there are a number of possible approaches to this problem. For tsunamis, the solution must be subject to the condition (18) and be matched to (19) and (17) through the ocean floor boundary conditions (16).

One possibly important effect of a realistic solid earth model is the influence of a sedimentary layer at the ocean bottom. This problem has been examined, with regard to *propagation*, by Okal (1982) and shown to be negligible for practical cases. Low rigidity of the ocean floor may be important for tsunami *excitation* by very shallow sources, however.

Another natural extension of our model is the inclusion of finite earthquake sources. This has been partially developed by Comer (1982), and applied to an analysis of real tide gauge data. Such further developments and applications will be considered in future papers. Also, a direct application of the point source solution presented here, a check on the general validity of the traditional, partially coupled tsunami generation models, is the subject of a companion paper (Comer 1984).

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